

Oblique Circular Cones and Cylinders

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ABSTRACT. Surface area and mean width of a cylinder (the convex hull of two parallel disks) in \mathbb{R}^3 are computed. It is more difficult to obtain analogous results for a cone (the convex hull of a disk D and a point p). Oblique formulas for mean width, as well as those for mean curvature, are new. Let ℓ denote the unique diameter of D whose endpoints are equidistant from p . We conclude with a question involving the plane that bisects the cone and contains $\{p, \ell\}$, as p varies. What is the minimum ratio of the smaller measure to the larger?

Certain mathematical formulas need better marketing. An example is the formula for the surface area of an oblique circular cone. Although an exact expression has been known since at least 1825, it has been rediscovered several times by independent researchers. The same could be said for the surface area of an oblique circular cylinder, but the redundancy of effort here is less costly (the formula follows swiftly from a well-known expression for the circumference of an ellipse).

This paper continues our work [1] on convex hulls of two disks in \mathbb{R}^3 . Three numerical characteristics of a convex hull – volume, surface area and mean width – are central. These quantities, along with the Euler characteristic, form a basis of the space of all additive continuous measures that are invariant under rigid motions. “The mean width is a new measure on three-dimensional solids that enjoys equal rights with volume and surface area” [2]. As far as we know, our expressions for mean width in the oblique case have not appeared before.

If we split an oblique cylinder along its axis, the two resultant half-cylinders have equal VL , AR and MW . The same symmetry does not hold for an oblique cone. While $VL_1 = VL_2$ for the half-cones, the associations between $AR_1 \leq AR_2$ and $MW_1 \leq MW_2$ are more complicated. An optimization problem involving the ratios AR_1/AR_2 and MW_1/MW_2 will occupy us at the end. We agree to split a cone so that its cross-section is an isosceles triangle (uniquely); a removal of this requirement is left open for someone else to address. There is an issue of whether we employ total surface area (including bases) or lateral surface area (curved portions only). The parameters minimizing AR_1/AR_2 and MW_1/MW_2 are nontrivial, as will be seen.

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1. OBLIQUE CYLINDER

The solid cylinder Ω is the convex hull of the following two parallel disks in \mathbb{R}^3 :

$$\{(x, y, z) : x^2 + y^2 \leq 1 \text{ \& } z = 0\} \quad \text{and} \quad \{(x, y, z) : x^2 + (y - a)^2 \leq 1 \text{ \& } z = b\}$$

where WLOG $a \geq 0$, $b > 0$. It is easy to find the volume of Ω (integrating the area of horizontal slices):

$$VL = \int_0^b \pi \, dz = \pi b.$$

The curved portion of the boundary $\partial\Omega$ in the half-space $x \geq 0$ can be represented parametrically:

$$x = \sqrt{1 - v^2}, \quad y = au + v, \quad z = bu, \quad 0 \leq u \leq 1 \text{ \& } -1 \leq v \leq 1.$$

Let $x_u, x_v, y_u, y_v, z_u, z_v$ denote partial derivatives of x, y, z . Defining

$$E = (x_u, y_u, z_u) \cdot (x_u, y_u, z_u), \quad G = (x_v, y_v, z_v) \cdot (x_v, y_v, z_v),$$

$$F = (x_u, y_u, z_u) \cdot (x_v, y_v, z_v)$$

we have

$$\begin{aligned} AR &= \int_{\partial\Omega} dS = 2\pi + 2 \int_0^1 \int_{-1}^1 \sqrt{EG - F^2} \, dv \, du \\ &= 2\pi + 2 \int_0^1 \int_{-1}^1 \sqrt{\frac{a^2 v^2 + b^2}{1 - v^2}} \, dv \, du \\ &= 2 \left(\pi + 2\sqrt{a^2 + b^2} E \left(\frac{a^2}{a^2 + b^2} \right) \right) \end{aligned}$$

where

$$E(\mu) = \int_0^{\pi/2} \sqrt{1 - \mu \sin^2(\theta)} \, d\theta = \int_0^1 \sqrt{\frac{1 - \mu t^2}{1 - t^2}} \, dt$$

is the complete elliptic integral of the second kind.

To compute MW (via the “indirect approach” in [1]), first find the mean curvature H and integrate. Defining

$$\mathcal{N} = -\frac{(x_u, y_u, z_u) \times (x_v, y_v, z_v)}{|(x_u, y_u, z_u) \times (x_v, y_v, z_v)|} = \left(b\sqrt{\frac{1 - v^2}{a^2 v^2 + b^2}}, \frac{bv}{\sqrt{a^2 v^2 + b^2}}, \frac{-av}{\sqrt{a^2 v^2 + b^2}} \right),$$

$$L = (x_u, y_u, z_u) \cdot \mathcal{N}_u, \quad N = (x_v, y_v, z_v) \cdot \mathcal{N}_v,$$

$$M = \frac{1}{2} ((x_u, y_u, z_u) \cdot \mathcal{N}_v + (x_v, y_v, z_v) \cdot \mathcal{N}_u)$$

we have

$$\begin{aligned} \int_{\partial\Omega} H \, dS &= \int_{\partial\Omega} \frac{EN - 2FM + GL}{2(EG - F^2)} \, dS \\ &= 2 \int_0^1 \int_{-1}^1 \frac{EN - 2FM + GL}{2(EG - F^2)} \sqrt{EG - F^2} \, dv \, du \\ &= 2 \int_0^1 \int_{-1}^1 \frac{(a^2 + b^2)b}{2(a^2v^2 + b^2)^{3/2}} \sqrt{\frac{a^2v^2 + b^2}{1 - v^2}} \, dv \, du \\ &= \int_{-1}^1 \frac{(a^2 + b^2)b}{a^2v^2 + b^2} \frac{1}{\sqrt{1 - v^2}} \, dv = \pi \sqrt{a^2 + b^2}. \end{aligned}$$

Second, find the exterior dihedral angle along each edge and integrate. For example, the semicircular edge $x^2 + y^2 = 1$, $z = 0$ & $x \geq 0$ corresponds to $u = 0$ & $-1 \leq v \leq 1$. Call this ε . The exterior dihedral angle is

$$\alpha = \arccos(\mathcal{N} \cdot (0, 0, -1)) = \arccos\left(\frac{av}{\sqrt{a^2v^2 + b^2}}\right)$$

and arclength s satisfies

$$ds = \sqrt{x_v^2 + y_v^2} \, dv = \frac{1}{\sqrt{1 - v^2}} \, dv.$$

It follows that

$$\int_{\varepsilon} \alpha \, ds = \int_{-1}^1 \frac{1}{\sqrt{1 - v^2}} \arccos\left(\frac{av}{\sqrt{a^2v^2 + b^2}}\right) \, dv = \frac{1}{2} \pi^2.$$

Finally, the surface $\partial\Omega$ is piecewise continuously differentiable and has $n = 4$ smooth edges ε_j with (non-constant) dihedral angles α_j , $1 \leq j \leq n$. From the general formula

$$MW = \frac{1}{2\pi} \int_{\partial\Omega} H \, dS + \frac{1}{4\pi} \sum_{j=1}^n \int_{\varepsilon_j} \alpha_j \, ds,$$

we deduce that

$$MW = \frac{1}{2\pi} \left(\pi \sqrt{a^2 + b^2} \right) + \frac{1}{4\pi} \left(\frac{4}{2} \pi^2 \right) = \frac{1}{2} \left(\sqrt{a^2 + b^2} + \pi \right).$$

In the special case $a = 0$ (a right circular cylinder), $MW = (b + \pi)/2$, which is consistent with [3, 4].

1.1. Comment about Ellipses. It is well known [5, 6, 7] that the lateral surface area of a cylinder is the product of an element length, $\sqrt{a^2 + b^2}$, and the circumference of a perpendicular section. The latter is an ellipse with semi-major axis 1 and semi-minor axis $b/\sqrt{a^2 + b^2}$. This is the shadow cast by the unit xy -circle into any plane (in xyz -space) normal to the vector $(0, a, b)$. In particular, the length of the projection of vector $(0, 1, 0)$ onto vector $(0, b, -a)$ is

$$\frac{(0, 1, 0) \cdot (0, b, -a)}{\sqrt{a^2 + b^2}} = \frac{b}{\sqrt{a^2 + b^2}}$$

which is the desired semi-minor axis. The eccentricity squared is hence

$$e^2 = 1 - \frac{b^2}{a^2 + b^2} = \frac{a^2}{a^2 + b^2}$$

and the circumference of the ellipse is $4E(e^2)$.

Older references discussing the surface area of a cylinder include [8, 9, 10]. A new result [11] provides an AGM-like iteration that permits rapid calculation of $E(e^2)$.

2. OBLIQUE CONE

The solid cone Ω is the convex hull of a disk and a point in \mathbb{R}^3 :

$$\{(x, y, z) : x^2 + y^2 \leq 1 \text{ \& } z = 0\} \quad \text{and} \quad (0, a, b)$$

where WLOG $a \geq 0, b > 0$. It is easy, as before, to find the volume of Ω :

$$VL = \int_0^b \pi \left(1 - \frac{z}{b}\right)^2 dz = -\frac{1}{3} \pi \left(1 - \frac{z}{b}\right)^3 b \Big|_0^b = \frac{1}{3} \pi b.$$

The curved portion of the boundary $\partial\Omega$ in the half-space $x \geq 0$ can be represented parametrically:

$$x = (1 - u)\sqrt{1 - v^2}, \quad y = au + (1 - u)v, \quad z = bu, \quad 0 \leq u \leq 1 \text{ \& } -1 \leq v \leq 1.$$

After computing E , F , G , we have

$$\begin{aligned}
AR &= \int_{\partial\Omega} dS = \pi + 2 \int_0^1 \int_{-1}^1 \sqrt{EG - F^2} dv du \\
&= \pi + 2 \int_0^1 \int_{-1}^1 (1-u) \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv du \\
&= \pi + 2\sqrt{s_0 s_1} [E(c_0) - K(c_0) + (1-c_1)\Pi(c_1, c_0)]
\end{aligned}$$

where

$$\begin{aligned}
s_0 &= \sqrt{(1-a)^2 + b^2}, & s_1 &= \sqrt{(1+a)^2 + b^2}, \\
c_0 &= \frac{1}{2} \left(1 - \frac{1-a^2+b^2}{s_0 s_1} \right), & c_1 &= \frac{1}{2} \left(1 - \frac{1+a^2+b^2}{s_0 s_1} \right).
\end{aligned}$$

In the preceding,

$$K(\mu) = \int_0^{\pi/2} \frac{1}{\sqrt{1-\mu \sin(\theta)^2}} d\theta = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-\mu t^2)}} dt$$

is the complete elliptic integral of the first kind and

$$\Pi(\nu, \mu) = \int_0^{\pi/2} \frac{1}{(1-\nu \sin(\theta)^2) \sqrt{1-\mu \sin(\theta)^2}} d\theta = \int_0^1 \frac{1}{(1-\nu t^2) \sqrt{(1-t^2)(1-\mu t^2)}} dt$$

is the complete elliptic integral of the third kind. Interestingly, a rapid AGM-like iteration to calculate $\Pi(c_1, c_0)$ would seem still to be awaiting discovery. This would “finish the quest” because such iterations for $E(c_0)$ and $K(c_0)$ are already in our possession [11].

As before,

$$\begin{aligned}
\mathcal{N} &= -\frac{(x_u, y_u, z_u) \times (x_v, y_v, z_v)}{|(x_u, y_u, z_u) \times (x_v, y_v, z_v)|} \\
&= \left(b \sqrt{\frac{1-v^2}{(1-av)^2 + b^2}}, \frac{bv}{\sqrt{(1-av)^2 + b^2}}, \frac{1-av}{\sqrt{(1-av)^2 + b^2}} \right)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\partial\Omega} H dS &= 2 \int_0^1 \int_{-1}^{-1} \frac{EN - 2FM + GL}{2(EG - F^2)} \sqrt{EG - F^2} dv du \\
&= 2 \int_0^1 \int_{-1}^{-1} \frac{(1 + a^2 + b^2 - 2av)b}{2(1 - u)((1 - av)^2 + b^2)^{3/2}} \\
&\quad \cdot (1 - u) \sqrt{\frac{(1 - av)^2 + b^2}{1 - v^2}} dv du \\
&= \int_{-1}^1 \frac{(1 + a^2 + b^2 - 2av)b}{(1 - av)^2 + b^2} \frac{1}{\sqrt{1 - v^2}} dv \\
&= \frac{1}{2} \left(\sqrt{a^2 + (-i + b)^2} + \sqrt{a^2 + (i + b)^2} \right) \pi
\end{aligned}$$

where i is the imaginary unit and we take the branch cut along the negative real axis (for both square root and logarithm functions).

Let ε denote the arc of the semicircle $x^2 + y^2 = 1$, $z = 0$ & $x \geq 0$. The exterior dihedral angle is

$$\alpha = \arccos(\mathcal{N} \cdot (0, 0, -1)) = \arccos\left(\frac{-1 + av}{\sqrt{(1 - av)^2 + b^2}}\right)$$

and arclength s satisfies

$$ds = \sqrt{x_v^2 + y_v^2} dv = \frac{1}{\sqrt{1 - v^2}} dv.$$

It follows that

$$\begin{aligned}
\int_{\varepsilon} \alpha ds &= \int_{-1}^1 \frac{1}{\sqrt{1 - v^2}} \arccos\left(\frac{-1 + av}{\sqrt{(1 - av)^2 + b^2}}\right) dv \\
&= \frac{1}{2} \left[\pi + i \ln\left(-i + b + \sqrt{a^2 + (-i + b)^2}\right) - i \ln\left(i + b + \sqrt{a^2 + (i + b)^2}\right) \right] \pi
\end{aligned}$$

and therefore

$$\begin{aligned}
MW &= \frac{1}{2\pi} \int_{\partial\Omega} H dS + \frac{2}{4\pi} \int_{\varepsilon} \alpha ds \\
&= \frac{1}{4} \left(\sqrt{a^2 + (-i + b)^2} + \sqrt{a^2 + (i + b)^2} \right) \\
&\quad + \frac{1}{4} \left[\pi + i \ln\left(-i + b + \sqrt{a^2 + (-i + b)^2}\right) - i \ln\left(i + b + \sqrt{a^2 + (i + b)^2}\right) \right] \pi.
\end{aligned}$$

In the special case $a = 0$ (a right circular cone),

$$\begin{aligned}
 MW &= \frac{1}{4} [(-i + b) + (i + b)] + \frac{1}{4} [\pi + i \ln(-2i + 2b) - i \ln(2i + 2b)] \\
 &= \frac{b}{2} + \frac{\pi}{4} + \frac{i}{4} [\ln(2) + \ln(i - b) - i\pi] - \frac{i}{4} [\ln(2) + \ln(i + b)] \\
 &= \frac{b}{2} + \frac{\pi}{2} - \frac{1}{2} \left[\frac{i}{2} \ln(i + b) - \frac{i}{2} \ln(i - b) \right] = \frac{b + \pi}{2} - \frac{1}{2} \arctan(b)
 \end{aligned}$$

which is consistent with [4].

2.1. Comment about Ruled Surfaces. The curved portion of a cone is a ruled surface, which means that through every point there exists a linear element that lies on the surface. A mistaken argument in [12] gave that the lateral surface area of a right circular cone is $2\pi\sqrt{1+b^2}$, which seems natural since the element length is $\sqrt{1+b^2}$ and the base circumference is 2π . This is an error: the correct value [5] should be $\pi\sqrt{1+b^2}$ (seen by setting $a = 0$, implying that $c_0 = c_1 = 0$, in our formula for AR). Alternatively, we might use Pappus's centroid theorem or a solid-of-revolution approach: letting $y = 1 - z/b$,

$$\begin{aligned}
 AR &= \int_0^b 2\pi y \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz = 2\pi \sqrt{1 + \frac{1}{b^2}} \int_0^b \left(1 - \frac{z}{b}\right) dz \\
 &= 2\pi \sqrt{\frac{1}{b^2} + 1} \cdot \left(-\frac{b}{2}\right) \left(1 - \frac{z}{b}\right)^2 \Big|_0^b = \pi \sqrt{1 + b^2}.
 \end{aligned}$$

The mistake was generalized to oblique circular cones in [12], under the hope that integrating the (varying) element length would provide the surface area. This too is an error: the correct procedure is found in [6]. Also, the double angle formula for cosine was misapplied, hence the last coefficient “2” in formula (6) of [12] should be “4” (and likewise in subsequent formulas).

Legendre [13] was the first person to explicitly give AR in terms of elliptic integrals. Follow-up work appeared in [14, 15, 16, 17, 18, 19, 20, 21, 22]. Older references discussing the surface area of a cone include [9, 23, 24, 25, 26, 27]. The oblique formula for AR was publicized early in the *Edinburgh Encyclopedia* [28], *Encyclopedia Britannica* [29] and elsewhere [30, 31, 32, 33]; thus the marketing fiasco has occurred only relatively recently. Legendre's achievement deserves not to be forgotten!

3. OPTIMIZATION PRELIMINARIES

Considering all cones Ω of equal volume, which one has the least surface area? An elementary solution (without use of elliptic integrals) is obtained by differentiating

$$AR - \pi = \int_{-1}^1 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv$$

twice with respect to a :

$$\begin{aligned} \frac{\partial}{\partial a} \int_{-1}^1 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv &= \int_{-1}^1 \frac{-v(1-av)}{\sqrt{(1-av)^2 + b^2}} \frac{1}{\sqrt{1-v^2}} dv, \\ \frac{\partial^2}{\partial a^2} \int_{-1}^1 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv &= \int_{-1}^1 \frac{b^2 v^2}{[(1-av)^2 + b^2]^{3/2}} \frac{1}{\sqrt{1-v^2}} dv. \end{aligned}$$

Setting $a = 0$ in the former expression, an odd function appears inside the integral, hence the first derivative vanishes. The second derivative is always positive, ensuring that the right circular cone is minimal. Our approach follows [34, 35], but caution should be exercised: beginning with formula (2.1) of [34], $a^2 + h^2 + r^2 - 2ar \cos(\theta)$ should be everywhere replaced by $a^2 \cos(\theta)^2 + h^2 + r^2 - 2ar \cos(\theta)$. The same error occurs in [35]. Again, see [6] for supporting details.

Considering all cones Ω of equal volume, which one has the least mean width? Starting with the algebraic expression for MW , it is easily verified that the right circular cone is minimal here too.

Before turning to more complicated optimization problems, let us define $\xi(x)$ for any $x > 0$ to be the largest real zero y of $xy - \cos(y)$. The most famous value of this function is

$$\xi(1) = 0.7390851332151606416553120\dots,$$

popularized in [36]. Define also $\eta(x)$ for any $0 < x < 1$ to be the largest real zero y of $xy - \sin(y)$. We will require two values:

$$\begin{aligned} \xi\left(\frac{2}{\pi}\right) &= 0.9340137863539518545607006\dots, \\ \eta\left(\frac{1}{\pi}\right) &= 2.3137341320786811322489898\dots \end{aligned}$$

In preparing the following solutions, we are being somewhat hasty. Rigorous proofs of minimality are not found; high-precision numerical confirmations often constitute the basis of our intuition.

4. RATIO OF SURFACE AREAS

Exact expressions for the two integrals:

$$\int_{-1}^0 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv \quad \text{and} \quad \int_0^1 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv$$

seem to be infeasible, despite the fact that we know already their *sum* and the availability of various transformations [37]. This will not be a deterrent, however. because only the case $a > 1$ & $b \rightarrow 0^+$ is conjectured to be relevant.

4.1. Lateral AR. Since $a > 1$, we have

$$\begin{aligned} \lim_{b \rightarrow 0^+} \frac{\int_0^1 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv}{\int_{-1}^0 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv} &= \frac{\int_0^{1/a} \frac{1-av}{\sqrt{1-v^2}} dv - \int_{1/a}^1 \frac{1-av}{\sqrt{1-v^2}} dv}{\int_{-1}^0 \frac{1-av}{\sqrt{1-v^2}} dv} \\ &= \frac{\pi - 2a + 4\sqrt{a^2 - 1} - 4 \operatorname{arcsec}(a)}{\pi + 2a}. \end{aligned}$$

After differentiating, the equation to be solved is

$$\frac{\pi}{2a} \sqrt{a^2 - 1} = \operatorname{arccsc}(a)$$

hence

$$a = 1.2437608987462040336147443....$$

Further,

$$\sin\left(\frac{\pi}{2a} \sqrt{a^2 - 1}\right) = \frac{1}{a}$$

which (after defining x to be the argument of \sin) implies

$$\frac{\pi}{\sqrt{\pi^2 - 4x^2}} \sin(x) = 1$$

which implies

$$\pi^2 \sin(x)^2 = \pi^2 - 4x^2$$

which implies

$$4x^2 = \pi^2 \cos(x)^2$$

which implies

$$2x = \pi \cos(x)$$

hence $x = \xi(2/\pi)$. Thus the infimum of ratios is 0.1892....

4.2. Total AR. Since $a > 1$, we have

$$\begin{aligned} \lim_{b \rightarrow 0^+} \frac{\frac{\pi}{2} + \int_0^1 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv}{\frac{\pi}{2} + \int_{-1}^0 \sqrt{\frac{(1-av)^2 + b^2}{1-v^2}} dv} &= \frac{\frac{\pi}{2} + \int_0^{1/a} \frac{1-av}{\sqrt{1-v^2}} dv - \int_{1/a}^1 \frac{1-av}{\sqrt{1-v^2}} dv}{\frac{\pi}{2} + \int_{-1}^0 \frac{1-av}{\sqrt{1-v^2}} dv} \\ &= \frac{-a + 2\sqrt{a^2-1} + 2 \operatorname{arccsc}(a)}{\pi + a}. \end{aligned}$$

After differentiating, the equation to be solved is

$$\frac{\pi}{a} \left(a - \sqrt{a^2-1} \right) = \operatorname{arcsec}(a)$$

hence

$$a = 1.4782960807222794430758369....$$

Further,

$$\cos \left(\frac{\pi}{a} \left(a - \sqrt{a^2-1} \right) \right) = \frac{1}{a}$$

which (after defining x to be the argument of \cos) implies

$$\frac{\pi}{\sqrt{2\pi x - x^2}} \cos(x) = 1$$

which implies

$$\pi^2 \cos(x)^2 = 2\pi x - x^2$$

which implies

$$(\pi - x)^2 = \pi^2 - 2\pi x + x^2 = \pi^2 - \pi^2 \cos(x)^2 = \pi^2 \sin(x)^2$$

which implies

$$\pi - x = \pi \sin(x)$$

which (after defining $w = \pi - x$) implies

$$w = \pi \sin(\pi - w) = \pi \sin(w)$$

hence $w = \eta(1/\pi)$. Thus the infimum of ratios is 0.4729....

5. RATIO OF MEAN WIDTHS

Unlike the preceding, exact expressions for the two integrals:

$$J_0 = \int_{-1}^0 \frac{(1 + a^2 + b^2 - 2av)b}{(1 - av)^2 + b^2} \frac{1}{\sqrt{1 - v^2}} dv,$$

$$J_1 = \int_0^1 \frac{(1 + a^2 + b^2 - 2av)b}{(1 - av)^2 + b^2} \frac{1}{\sqrt{1 - v^2}} dv$$

are needed. Only the case $a > 1$ & $b \rightarrow 0^+$ is conjectured to be relevant. Integral J_1 is problematic due to the singularity at $v = 1/a$ as $b \rightarrow 0^+$, whereas integral $J_0 \rightarrow 0$ clearly.

5.1. Integrated Mean Curvature. It is not difficult to show that

$$J_0 = \frac{i}{2} \left\{ \sqrt{a^2 + (-i + b)^2} \left[\frac{i\pi}{2} - \ln(i - b) + \ln \left(a + \sqrt{a^2 + (-i + b)^2} \right) \right] \right. \\ \left. + \sqrt{a^2 + (i + b)^2} \left[\frac{i\pi}{2} + \ln(-i - b) - \ln \left(a + \sqrt{a^2 + (i + b)^2} \right) \right] \right\}$$

From the known expression for $J_0 + J_1$, we deduce that

$$J_1 = \frac{i}{2} \left\{ \sqrt{a^2 + (-i + b)^2} \left[-\frac{3i\pi}{2} + \ln(i - b) - \ln \left(a + \sqrt{a^2 + (-i + b)^2} \right) \right] \right. \\ \left. + \sqrt{a^2 + (i + b)^2} \left[-\frac{3i\pi}{2} - \ln(-i - b) + \ln \left(a + \sqrt{a^2 + (i + b)^2} \right) \right] \right\}$$

and therefore $J_1 \rightarrow \pi\sqrt{a^2 - 1}$.

5.2. Exterior Dihedral Angles. The isosceles triangle contains vertices $(0, a, b)$, $(-1, 0, 0)$, $(1, 0, 0)$, hence is contained in the plane $-by + az = 0$. The exterior normal vector to this face depends on the choice of half-cone. The base of the triangle is of length 2; the sum of the lengths of the two triangular legs is $2\sqrt{a^2 + b^2 + 1}$.

For the lower (smaller) half-cone, the exterior normal vector is $(0, -b, a)$. At the base edge, the dihedral angle is

$$\arccos \left((0, 0, -1) \cdot \frac{(0, -b, a)}{\sqrt{a^2 + b^2}} \right) = \pi - \arccos \left(\frac{a}{\sqrt{a^2 + b^2}} \right)$$

and at each leg edge (for which $v = 0$), the dihedral angle is

$$\arccos \left(\mathcal{N} \cdot \frac{(0, -b, a)}{\sqrt{a^2 + b^2}} \right) = \arccos \left(\frac{a}{\sqrt{1 + b^2} \sqrt{a^2 + b^2}} \right).$$

For the upper (larger) half-cone, the exterior normal vector is $(0, b, -a)$. At the base edge, the dihedral angle is

$$\arccos \left((0, 0, -1) \cdot \frac{(0, b, -a)}{\sqrt{a^2 + b^2}} \right) = \arccos \left(\frac{a}{\sqrt{a^2 + b^2}} \right)$$

and at each leg edge, the dihedral angle is

$$\arccos \left(\mathcal{N} \cdot \frac{(0, b, -a)}{\sqrt{a^2 + b^2}} \right) = \pi - \arccos \left(\frac{a}{\sqrt{1 + b^2} \sqrt{a^2 + b^2}} \right).$$

Note also that

$$L_0 = 2 \int_{-1}^0 \frac{1}{\sqrt{1 - v^2}} \arccos \left(\frac{-1 + av}{\sqrt{(1 - av)^2 + b^2}} \right) dv \rightarrow 2 \int_{-1}^0 \frac{\pi}{\sqrt{1 - v^2}} dv = \pi^2,$$

$$L_1 = 2 \int_0^1 \frac{1}{\sqrt{1 - v^2}} \arccos \left(\frac{-1 + av}{\sqrt{(1 - av)^2 + b^2}} \right) dv \rightarrow 2 \int_0^{1/a} \frac{\pi}{\sqrt{1 - v^2}} dv = 2\pi \operatorname{arccsc}(a)$$

as $b \rightarrow 0^+$, for upper/lower semicircular edges.

5.3. Culmination. As $b \rightarrow 0^+$, the ratio

$$\frac{\frac{J_1}{2\pi} + \frac{1}{4\pi} \left[L_1 + 2 \left(\pi - \arccos \left(\frac{a}{\sqrt{a^2 + b^2}} \right) \right) + 2\sqrt{a^2 + b^2 + 1} \arccos \left(\frac{a}{\sqrt{1 + b^2} \sqrt{a^2 + b^2}} \right) \right]}{\frac{J_0}{2\pi} + \frac{1}{4\pi} \left[L_0 + 2 \arccos \left(\frac{a}{\sqrt{a^2 + b^2}} \right) + 2\sqrt{a^2 + b^2 + 1} \left(\pi - \arccos \left(\frac{a}{\sqrt{1 + b^2} \sqrt{a^2 + b^2}} \right) \right) \right]}$$

approaches

$$\frac{\frac{1}{2}\sqrt{a^2 - 1} + \frac{1}{2} \operatorname{arccsc}(a) + \frac{1}{2}}{\frac{\pi}{4} + \frac{1}{2}\sqrt{a^2 + 1}}.$$

After differentiating, the equation to be solved is

$$\frac{\sqrt{a^2 - 1} (2 + \pi\sqrt{a^2 + 1})}{2a^2} = 1 + \operatorname{arccsc}(a)$$

hence

$$a = 1.3638337555895594010839152....$$

Thus the infimum of ratios is 0.8431....

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7. ADDENDUM

There is a curious asymmetry in our discussions of AR in Section 4 and of MW in Section 5. For AR , we imagine a fixed amount of sheet metal, allocated between the two half-cones (thus $AR_1 + AR_2$ is equal to the original AR). In contrast, for MW , we treat the half-cones as independent convex solids – think of widths as drawn from wholly separate objects – although each object contains the isosceles triangle as a face (thus $MW_1 + MW_2$ is larger than the original MW).

To make things compatible, let us revisit Section 4.2, but now include the overlap. The triangular area is $\frac{1}{2} \cdot 2 \cdot \sqrt{a^2 + b^2}$. Hence the new ratio is

$$\lim_{b \rightarrow 0^+} \frac{\sqrt{a^2 + b^2} + \frac{\pi}{2} + \int_0^1 \sqrt{\frac{(1 - av)^2 + b^2}{1 - v^2}} dv}{\sqrt{a^2 + b^2} + \frac{\pi}{2} + \int_{-1}^0 \sqrt{\frac{(1 - av)^2 + b^2}{1 - v^2}} dv} = \frac{2\sqrt{a^2 - 1} + 2 \operatorname{arccsc}(a)}{\pi + 2a}.$$

After differentiating, the new equation to be solved is

$$\frac{\pi}{2a} \sqrt{a^2 - 1} = \operatorname{arccsc}(a)$$

which is identical to the old one in Section 4.1; therefore

$$a = 1.2437608987462040336147443...$$

reappears here. The objective function is different, however, and the infimum of ratios is 0.5946....

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